

CAPACITIES AND HAUSDORFF MEASURES ON METRIC SPACES

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ABSTRACT. In this article, we show that in a Q -doubling space (X, d, μ) , $Q > 1$, that supports a Q -Poincaré inequality and satisfies a chain condition, sets of Q -capacity zero have generalized Hausdorff h -measure zero for $h(t) = \log^{1-Q-\epsilon}(1/t)$.

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1. INTRODUCTION

The relation between capacities and generalized Hausdorff measures in \mathbb{R}^n and in metric spaces has been studied for many years. In \mathbb{R}^n , it is known that sets of p -capacity zero have generalized Hausdorff h -measure zero provided that

$$(1.1) \quad \int_0^1 (t^{p-n} h(t))^{\frac{1}{p-1}} \frac{dt}{t} < \infty,$$

for $1 < p \leq n$, see Theorem 7.1 in [KM72] or Theorem 5.1.13 in [AH96]. In particular, the Hausdorff dimension of such sets does not exceed $n - p$. Similar results for weighted capacities and Hausdorff measures in \mathbb{R}^n can be found e.g. in [HKM06].

Let us consider a doubling metric space (X, d, μ) . Then a simple iteration argument shows that there is an exponent $Q > 0$ and a constant $C \geq 1$ so that

$$(1.2) \quad \left(\frac{s}{r}\right)^Q \leq C \frac{\mu(B(x, s))}{\mu(B(a, r))}$$

holds whenever $a \in X$, $x \in B(a, r)$ and $0 < s \leq r$. We say that (X, d, μ) is Q -doubling if (X, d, μ) is a doubling metric measure space and (1.2) holds with the given Q . Towards defining our Sobolev space, we recall that a measurable function $g \geq 0$ is an upper gradient of a measurable function u provided

$$(1.3) \quad |u(\gamma(a)) - u(\gamma(b))| \leq \int_{\gamma} g \, ds$$

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for every rectifiable curve $\gamma : [a, b] \rightarrow X$ [HK98], [KM98]. We define $W^{1,p}(X)$, $1 \leq p < \infty$, to be the collection of all $u \in L^p(X)$ that have an upper gradient that also belongs to $L^p(X)$, see [Sha00]. In order to obtain lower bounds for the capacity associated to $W^{1,p}(X)$, it suffices to assume a suitable Poincaré inequality. We say that (X, d, μ) supports a p -Poincaré inequality if there exist constants C and λ such that

$$(1.4) \quad \int_B |u - u_B| d\mu \leq C \operatorname{diam}(B) \left(\int_{\lambda B} g^p d\mu \right)^{1/p}$$

for every open ball B in X , for every function $u : X \rightarrow \mathbb{R}$ that is integrable on balls, and for every upper gradient g of u in X . For simplicity, we will from now on only consider the case of a Q -doubling space and we will assume that $p = Q$.

In this paper, we study the relation between Q -capacity and generalized Hausdorff h -measure for $h(t) = \log^{1-Q-\epsilon}(1/t)$ (see Section 2 for the definitions of capacity and Hausdorff h -measure) on a Q -doubling metric measure space that supports a Q -Poincaré inequality. Björn and Onninen proved in [BO05] that a compact set K in a Q -doubling space that supports a 1-Poincaré inequality has Hausdorff h -measure zero provided that Q -capacity of K is zero, for any h that satisfies (1.1) with n replaced by Q . Hence this holds for $h(t) = \log^{1-Q-\epsilon}(1/t)$ for any $\epsilon > 0$. Under the weaker assumption of a Q -Poincaré inequality, their work shows that K has Hausdorff h -measure zero, for $h(t) = \log^{-Q-\epsilon}(1/t)$. They pose an open problem that in our setting asks if the above analogue of (1.1) is sufficient for h even under a Q -Poincaré inequality assumption. An examination of the corresponding proof in [BO05] shows that it actually suffices that the Poincaré inequality (1.4) holds for each $u \in W^{1,Q}(X)$ with $p = 1$ for some function $g \in L^Q(X)$, whose Q -norm is at most a fixed constant times the infimum of Q -norms of all upper gradients of u . This requirement holds for complete Q -doubling spaces that supports a Q -Poincaré inequality by the self-improving property of Poincaré inequalities [KZ08], for details see Section 4 of [KK]. However, the self-improving property from [KZ08] may fail in the non-complete setting, see [Kos99].

We establish the optimal result for logarithmic gauge functions h under a mild additional assumption.

Theorem 1.1. *Let $\epsilon > 0$. Let (X, μ) be a Q -doubling space for some $Q > 1$ that supports a Q -Poincaré inequality and assume that X satisfies a chain condition (see definition 3.1).*

Let $x_0 \in X$ and $R > 0$. Then we have $H^h(E) = 0$ for every compact $E \subset B(x_0, R)$ with $\text{cap}_Q(E, B(x_0, 2R)) = 0$, where $h(t) = \log^{1-Q-\epsilon}(1/t)$.

A doubling space that supports a p -Poincaré inequality is necessarily connected and even bi-Lipschitz equivalent to a geodesic space, if it is complete [Che99]. Since each geodesic space satisfies a chain condition, the assumption of chain condition in Theorem 1.1 is natural.

2. NOTATION AND PRELIMINARIES

We assume throughout that $X = (X, d, \mu)$ is a metric measure space equipped with a metric d and a Borel regular outer measure μ . We call such a μ a measure. The Borel-regularity of the measure μ means that all Borel sets are μ -measurable and that for every set $A \subset X$ there is a Borel set D such that $A \subset D$ and $\mu(A) = \mu(D)$.

We denote open balls in X with a fixed center $x \in X$ and radius $0 < r < \infty$ by

$$B(x, r) = \{y \in X : d(y, x) < r\}.$$

If $B = B(x, r)$ is a ball, with center and radius understood, and $\lambda > 0$, we write

$$\lambda B = B(x, \lambda r).$$

With small abuse of notation we write $\text{rad}(B)$ for the radius of a ball B and we always have

$$\text{diam}(B) \leq 2 \text{rad}(B),$$

and the inequality can well be strict.

A Borel regular measure μ on a metric space (X, d) is called a *doubling measure* if every ball in X has positive and finite measure and there exist a constant $C \geq 1$ such that

$$\mu(B(x, 2r)) \leq C_\mu \mu(B(x, r))$$

for each $x \in X$ and $r > 0$. We call a triple (X, d, μ) a *doubling metric measure space* if μ is a doubling measure on X .

If $A \subset X$ is a μ -measurable set with finite and positive measure, then the *mean value* of a function $u \in L^1(A)$ over A is

$$u_A = \int_A u d\mu = \frac{1}{\mu(A)} \int_A u d\mu.$$

A metric space is said to be *geodesic* if every pair of points in the space can be joined by a curve whose length is equal to the distance between the points.

Definition 2.1. Let $E \subset B(x_0, R)$ be compact. The Q -capacity of E with respect to the ball $B(x_0, 2R)$ is

$$\text{cap}_Q(E, B(x_0, 2R)) = \inf \|g\|_{L^Q(X)}$$

where the infimum is taken over all upper gradients g of all continuous functions u with compact support in $B(x_0, 2R)$ and $u \geq 1$ on E .

Let $h : [0, \infty) \rightarrow [0, \infty)$ be a non-decreasing function such that $\lim_{t \rightarrow 0+} h(t) = h(0) = 0$. For $0 < \delta \leq \infty$, and $E \subset X$, we define *generalized Hausdorff h -measure* by setting

$$H^h(E) = \limsup_{\delta \rightarrow 0} H_\delta^h(E),$$

where

$$H_\delta^h(E) = \inf \sum_i h(\text{diam}(B_i)),$$

where the infimum is taken over all collections of balls $\{B_i\}_{i=1}^\infty$ such that $\text{diam}(B_i) \leq \delta$ and $E \subset \bigcup_{i=1}^\infty B_i$. In particular, if $h(t) = t^\alpha$ with some $\alpha > 0$, then H^h is the usual α -dimensional Hausdorff measure, denoted also by H^α . See [Rog98] for more information on the generalized Hausdorff measure. Recall that the *Hausdorff h -content* of a set E in a metric space is the number

$$\mathcal{H}_\infty^h(E) = \inf \sum_i h(\text{diam}(B_i)),$$

where the infimum is taken over all countable covers of the set E by balls B_i . Thus the h -content of E is less than, or equal to, the Hausdorff h -measure of E , and it is never

infinite for E bounded. However, the h -content of set is zero if and only if its Hausdorff h -measure is zero.

For the convenience of reader we state here a fundamental covering lemma (for a proof see [Fed69, 2.8.4-6] or [Zie89, Theorem 1.3.1]).

Lemma 2.2 (5B-covering lemma). *Every family \mathcal{F} of balls of uniformly bounded diameter in a metric space X contains a pairwise disjoint subfamily \mathcal{G} such that for every $B \in \mathcal{F}$ there exists $B' \in \mathcal{G}$ with $B \cap B' \neq \emptyset$ and $\text{diam}(B) < 2 \text{diam}(B')$. In particular, we have that*

$$\bigcup_{B \in \mathcal{F}} B \subset \bigcup_{B \in \mathcal{G}} 5B.$$

We mention a technical lemma from [KK] and we give a simple proof here.

Lemma 2.3. *Suppose $\{a_j\}_{j=0}^{\infty}$ is a sequence of non-negative real numbers such that $\sum_{j \geq 0} a_j < \infty$. Then*

$$\sum_{j \geq 0} \frac{a_j}{\left(\sum_{i \geq j} a_i\right)^{1-\delta}} \leq \frac{1}{\delta} \left(\sum_{j \geq 0} a_j\right)^{\delta} < \infty \quad \text{for any } 0 < \delta < 1.$$

Proof. Define

$$u(t) = \sum_{j \geq 0} a_j \chi_{[j, j+1)}(t)$$

for $t \geq 0$ and $v(x) = \int_x^{\infty} u(t) dt$ for $x \geq 0$. Then v is a Lipschitz function and

$$v(x) = \sum_{j \geq 0} a_j v_j(x),$$

where

$$v_j(x) = \begin{cases} 1 & \text{if } x < j, \\ j+1-x & \text{if } j \leq x < j+1, \\ 0 & \text{if } x \geq j+1. \end{cases}$$

Then we have the required estimate

$$\sum_{j \geq 0} \frac{a_j}{\left(\sum_{i \geq j} a_i\right)^{1-\delta}} \leq \int_0^{\infty} \frac{-v'(x) dx}{v(x)^{1-\delta}} = \frac{1}{\delta} \left(\sum_{j \geq 0} a_j\right)^{\delta} < \infty.$$

□

3. PROOF OF THEOREM 1.1

Before we go into the proof of Theorem 1.1, let us recall a definition of a *chain condition* from [KK], a version of which is already introduced in [HK00].

Definition 3.1. We say that a space X satisfies a *chain condition* if for every $\lambda \geq 1$ there are constants $M \geq 1$, $0 < m \leq 1$ such that for each $x \in X$ and all $0 < r < \text{diam}(X)/8$ there is a sequence of balls B_0, B_1, B_2, \dots with

1. $B_0 \subset X \setminus B(x, r)$,
2. $M^{-1} \text{diam}(B_i) \leq \text{dist}(x, B_i) \leq M \text{diam}(B_i)$,
3. $\text{dist}(x, B_i) \leq Mr2^{-mi}$,
4. there is a ball $D_i \subset B_i \cap B_{i+1}$, such that $B_i \cup B_{i+1} \subset MD_i$,
for all $i \in \mathbb{N} \cup \{0\}$ and
5. no point of X belongs to more than M balls λB_i .

The sequence B_i will be called a *chain associated with x, r* .

The existence of a doubling measure on X does not guarantee a chain condition. In fact, such a space can be badly disconnected, whereas a space with a chain condition cannot have “large gaps”. For example, the standard 1/3-Cantor set satisfies a chain condition only for $\lambda < 2$. On the other hand, geodesic and many other spaces satisfy our chain condition, see [KK]. We recall a lemma from [KK] and we omit the proof here.

Lemma 3.2. *Suppose that X satisfies a chain condition and let the sequence B_i be a chain associated with x, R_1, R_2 for $x \in X$ and $0 < R_1 < R_2 < \text{diam}(X)/4$. Then we can find balls $B_{i_{R_2}}, B_{i_{R_2}+1}, \dots, B_{i_{R_1}}$ from the above collection such that*

$$(3.1) \quad \frac{R_2}{M(1+M)^2} \leq \text{diam}(B_{i_{R_2}}) \leq MR_2,$$

$$(3.2) \quad \frac{R_1}{M(1+M)^2} \leq \text{diam}(B_{i_{R_1}}) \leq MR_1$$

hold and $B_{i_{R_2}} \subset B(x, R_2)$, $B_{i_{R_1}} \subset B(x, R_1)$ and also the balls $B_{i_{R_2}}, B_{i_{R_2}+1}, \dots, B_{i_{R_1}}$ form a chain.

Proof of Theorem 1.1. For notational simplicity, we assume $R = 1/8$. Let u be a continuous function with compact support in $B(x_0, 1/4)$ and $u \geq 1$ on E . Let g be an upper gradient of u . We construct

$$(3.3) \quad E_{\epsilon, M} = \left\{ x \in E : \exists \text{ some } r_x < 10 \text{ so that } \int_{B(x, r_x)} g^Q d\mu \geq M \log^{1-Q-\epsilon} \left(\frac{10}{r_x} \right) \right\},$$

M to be chosen later.

Let $x \in E \setminus E_{\epsilon, M}$. Let $k \in \mathbb{N}$. Then we apply Lemma 3.2 for $R_1 = 2^{-k}$, $R_2 = 2^{-1}$ to get a chain of balls B_1, B_2, \dots, B_{i_k} . Using the doubling property, Poincaré inequality and Lemma 3.2, we obtain

$$\begin{aligned} |u_{B_{i_k}} - u_{B(x, 2^{-k})}| &\leq \int_{B_{i_k}} |u - u_{B(x, 2^{-k})}| d\mu \\ &\leq c \int_{u_{B(x, 2^{-k})}} |u - u_{B(x, 2^{-k})}| d\mu \\ &\leq c \left(\int_{B(x, 2^{-k})} g^Q d\mu \right)^{\frac{1}{Q}} \rightarrow 0 \text{ as } k \rightarrow \infty \end{aligned}$$

and hence $u_{B_{i_k}} \geq 2/3$ for large k , by the continuity of u . We assume that $u_{B_1} \leq 1/3$, as we can always do it by increasing the radius R_2 .

Let $\tilde{\epsilon} > 0$, which is to be chosen later. We use a telescopic argument for the balls B_1, B_2, \dots, B_{i_k} and also use chain conditions, relative lower volume decay (1.2) and

Poincaré inequality (1.4) to obtain

$$\begin{aligned}
\frac{1}{3} \leq |u_{B_{i_k}} - u_{B_1}| &\leq \sum_{n=1}^{i_k-1} |u_{B_n} - u_{B_{n+1}}| \\
&\leq \sum_{n=1}^{i_k-1} (|u_{B_n} - u_{D_n}| + |u_{B_{n+1}} - u_{D_n}|) \\
&\leq \sum_{n=1}^{i_k} \left(\int_{D_n} |u - u_{B_n}| d\mu + \int_{D_n} |u - u_{B_{n+1}}| d\mu \right) \\
&\leq c \sum_{n=1}^{i_k} \int_{B_n} |u - u_{B_n}| d\mu \\
&\leq c \sum_{n=1}^{i_k} \text{diam}(B_n) \left(\int_{\lambda B_n} g^Q d\mu \right)^{\frac{1}{Q}} \\
&\leq c \sum_{n \geq 1} \left(\frac{\text{diam}(B_n)^Q}{\mu(B_n)} \int_{\lambda B_n} g^Q d\mu \right)^{\frac{1}{Q}} n^{\frac{Q-1+\tilde{\epsilon}}{Q}} n^{-\frac{Q-1+\tilde{\epsilon}}{Q}} \\
&\leq c \left(\sum_{n \geq 1} \frac{\text{diam}(B_n)^Q}{\mu(B_n)} n^{Q-1+\tilde{\epsilon}} \int_{\lambda B_n} g^Q d\mu \right)^{\frac{1}{Q}} \left(\sum_{n \geq 1} n^{-\frac{Q-1+\tilde{\epsilon}}{Q-1}} \right)^{\frac{Q-1}{Q}} \\
&\leq \frac{c}{\mu(B(x, 10))} \left(\sum_{n \geq 1} n^{Q-1+\tilde{\epsilon}} \int_{\lambda B_n} g^Q d\mu \right)^{\frac{1}{Q}}.
\end{aligned}$$

Since $x \in E \setminus E_{\epsilon, M}$, we have

$$(3.4) \quad \int_{B(x, r_x)} g^Q d\mu \leq M \log^{1-Q-\epsilon} \left(\frac{10}{r_x} \right)$$

for all $r_x < 10$. Hence we get

$$(3.5) \quad \sum_{m \geq n} \int_{\lambda B_m} g^Q d\mu \leq M n^{1-Q-\epsilon}$$

for all $n \geq 1$. Then we choose $\tilde{\epsilon} = \epsilon - \delta(Q - 1 - \epsilon)$ for some $0 < \delta < 1$ (we can choose δ as small as we want to make $\tilde{\epsilon}$ positive) to obtain

$$1 \leq \frac{cM^{\frac{1-\delta}{Q}}}{\mu(B(x, 10))} \left(\sum_{n \geq 1} \frac{\int_{\lambda B_n} g^Q d\mu}{\left(\sum_{m \geq n} \int_{\lambda B_m} g^Q \right)^{1-\delta}} \right)^{\frac{1}{Q}}.$$

Finally, we use Lemma 2.3 and (3.4) to get

$$\begin{aligned} 1 &\leq \frac{cM^{\frac{1-\delta}{Q}}}{\delta\mu(B(x, 10))} \left(\sum_{n \geq 1} \int_{\lambda B_n} g^Q d\mu \right)^{\frac{\delta}{Q}} \\ &\leq \frac{cM^{\frac{1}{Q}}}{\delta\mu(B(x, 10))}. \end{aligned}$$

If we choose $M < \delta^Q/c^Q$, where $0 < \delta < \epsilon/(Q-1+\epsilon)$, then we get contradiction in the above inequality to conclude that $E \setminus E_{\epsilon, M(\epsilon, Q, c)} = \emptyset$. In other words, for every $x \in E$, there exists $r_x < 10$ such that

$$\int_{B(x, r_x)} g^Q d\mu \geq M(\epsilon, Q, c) \log^{1-Q-\epsilon} \left(\frac{10}{r_x} \right).$$

By $5B$ -covering lemma, pick up a collection of disjoint balls $B_i = B(x_i, r_i)$ such that $E \subset \cup_i 5B_i$. Then

$$\begin{aligned} \int_X g^Q d\mu &\geq \sum_i \int_{B_i} g^Q d\mu \geq M(\epsilon, Q, c) \sum_i \log^{1-Q-\epsilon} \left(\frac{1}{r_i} \right) \\ &\geq M(\epsilon, Q, c) \log^{1-Q-\epsilon} \left(\frac{1}{\text{diam}(E)} \right), \end{aligned}$$

hence $\text{cap}_Q(E, B(x_0, 2R)) \geq M(\epsilon, Q, c) \mathcal{H}_\infty^h(E)$. \square

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